JOURNAL OF COMPUTATIONAL PHYSICS 20, 279–297 (1976)

Long Wavelength Approximations to Periodic Elastic Media and Numerical Applications

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Received April 28, 1975; revised October 6, 1975

The uniaxial motion of elastic media with rapidly varying spatial heterogeneities is investigated. Homogeneous dispersive media that provide long wavelength approximations to elastic media with periodic heterogeneities are developed. Applications to finite difference calculations in media whose properties vary rapidly within a wavelength are presented.

INTRODUCTION

In investigations of wave propagation phenomena, it is frequently desirable to solve the pertinent equations by time-marching numerical codes. Unfortunately, even for the simple case of one-dimensional motion in elastic media, the computations can become prohibitively expensive when rapidly varying spatial heterogeneities are present. To accurately reproduce the effects of heterogeneities on wave motion, a large number of zones per wavelength is required. As a result, the computations do not extend to distances much larger than a wavelength.

The main effect of elastic heterogeneities on wave motion is dispersion. However, the same kind of dispersion is generated in media that are homogeneous, provided their sound speeds are suitably chosen functions of frequency. Such media offer a numerical advantage because their motion can be calculated by zones whose dimensions are not limited by the dimensions of heterogeneities. In this paper, we examine uniaxial propagation in heterogeneous periodic elastic media. Through harmonic analysis, we show that at long wavelengths any periodic elastic medium behaves like a homogeneous dispersive medium. The simplest of these media has a constitutive relation that contains, in addition to the usual Hookean term, another term that, in the time domain, is proportional to the second time derivative of the strain. The transient properties of this medium are examined, and the stability of possible finite difference analogs is discussed. A particular finite difference

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analog, which is simple to implement in currently existing codes, but which, nevertheless, places undesirable limitations on the mesh variables, is investigated in some detail. An example comparing the exact transient motion developed in a periodic bilaminate to the motion obtained from the difference equations is presented.

Although the analysis is restricted to one-dimensional motion in periodic media, it seems that the method may also find applications in calculations of motion in media that are not periodic, for example nonperiodic heterogeneous media that look homogeneous and isotropic on a macroscopic scale.

Harmonic Analysis

Let the time harmonic dependence be $e^{-i\omega t}$, where ω is the applied frequency (a real number). Let x, $\hat{u}(x, \omega)$, $\hat{\sigma}(x, \omega)$ and $\hat{\epsilon}(x, \omega)$ represent the Lagrangian particle position, harmonic particle velocity, harmonic particle compressive stress, and harmonic particle strain for one-dimensional motion. Finally, let the referenced density $\rho(x)$, inverse constraint modulus $m(x)$ and reference sound speed $c(x)$, $c(x) = 1/(\rho(x)\, m(x))^{1/2}$, be positive piecewise-continuous periodic functions of x with period L. The uniaxial motion in heterogeneous elastic media is governed by the following equations relating these quantities

$$\partial\hat{u}/\partial x = -i\omega\hat{\epsilon}, \tag{1}$$

$$\partial\hat{\sigma}/\partial x = i\omega\rho(x)\,\hat{u}, \tag{2}$$

$$\hat{\sigma} = -\hat{\epsilon}/m(x). \tag{3}$$

The solution of the above equations has been investigated in [1], where it was shown that, with the possible exception of the frequencies at which

$$\sin[\omega L/v_p(\omega)] = 0,$$

the velocity and stress can be written as

$$\begin{bmatrix}\hat{u}(x, \omega)\\ \hat{\sigma}(x, \omega)\end{bmatrix} = f_+(\omega)\, e^{i(\omega x/v_p(\omega))} \begin{bmatrix} F_+(x, \omega) \\ Z_+(\omega)\; G_+(x, \omega)\end{bmatrix} + f_-(\omega)\, e^{-i(\omega x/v_p(\omega))} \begin{bmatrix} F_-(x, \omega) \\ Z_-(\omega)\; G_-(x, \omega)\end{bmatrix}, \tag{4}$$

where $F_\pm(x, \omega)$, $G_\pm(x, \omega)$ are periodic functions of x with period L and

$$F_\pm(nL, \omega) = G_\pm(nL, \omega) = 1, n = 0, 1, 2,.... \tag{5}$$

The functions $v_p(\omega)$, $Z_\pm(\omega)$, and $f_\pm(\omega)$, which are independent of x, represent, respectively, the wave phase velocity, forward and backward wave impedances, and particle velocity amplitudes generated by boundary conditions at $x = 0$.

The earlier investigation showed that the phase velocity and wave impedances can be obtained from a certain matrix $B(x, \omega)$,

$$B(x, \omega) = \begin{bmatrix} b_{11}(x, \omega) & b_{12}(x, \omega) \\ b_{21}(x, \omega) & b_{22}(x, \omega) \end{bmatrix},$$

which is the solution of the matrix differential equation

$$\frac{\partial B(x, \omega)}{\partial x} = i\omega \begin{bmatrix} 0 & m(x) \\ \rho(x) & 0 \end{bmatrix} B(x, \omega), \tag{6}$$

subject to the boundary condition

$$B(0, \omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

An examination of this matrix revealed that (a) its determinant is independent of x and equal to 1, (b) its diagonal elements $b_{11}(x, \omega)$ and $b_{22}(x, \omega)$ are always real functions, and (c) the off-diagonal elements $b_{12}(x, \omega)$ and $b_{21}(x, \omega)$ are purely imaginary. It turned out that at $x = L$, the eigenvalues of this matrix represent the wave propagation factors $e^{\pm i(\omega L/v_p(\omega))}$, whereas the corresponding eigenvectors $\left[\begin{smallmatrix} 1 \\ Z_\pm(\omega) \end{smallmatrix}\right]$ give the wave impedances. In particular, the phase velocity satisfies the dispersion relation

$$\cos\left[\frac{\omega L}{v_p(\omega)}\right] = \frac{b_{11}(L, \omega) + b_{22}(L, \omega)}{2}, \tag{7}$$

whereas the wave impedances are given by the equations

$$Z_\pm(\omega) = \frac{e^{\pm i(\omega L/v_p(\omega))} - b_{11}(L, \omega)}{b_{12}(L, \omega)} = \frac{b_{21}(L, \omega)}{e^{\pm i(\omega L/v_p(\omega))} - b_{22}(L, \omega)}. \tag{8}$$

Utilizing usual low frequency procedures, power series expansions for the elements of $B(L, \omega)$ were obtained. It was deduced from Eq. (7) that at long wavelengths (or low frequencies), the wave phase velocity is a real even function of ω and has the power series expansion

$$v_p = \bar{c}(1 - v_2\omega^2 - v_4\omega^4 - \cdots), \tag{9}$$

where the static speed $\bar{c}$ is given by

$$\bar{c} = \frac{1}{[(1/L)\int_0^L \rho(x)\,dx\,(1/L)\int_0^L m(x)\,dx]^{1/2}}. \tag{10}$$

The quadratic coefficient v_2, which is responsible for long wavelength dispersion, is related to the spatial variations of ρ and m as follows

$$v_2 = \frac{1}{2}\left(\frac{\bar{c}}{L}\right)^2 \left[\frac{1}{12}\left(\frac{L}{\bar{c}}\right)^4 - \int_0^L m(x) \int_0^x \rho(y) \int_0^y m(z) \int_0^z \rho(w)\, dw\, dz\, dy\, dx \right.$$
$$\left. - \int_0^L \rho(x) \int_0^x m(y) \int_0^y \rho(z) \int_0^z m(w)\, dw\, dz\, dy\, dx\right], \tag{11}$$

and is always positive if the medium impedance $\rho(x)\, c(x)$ depends on x (when the impedance is independent of x the phase velocity does not depend on ω and $v_2 = v_4 = \cdots = 0$). The higher-order coefficients, whose relation to ρ and m may be obtained by straightforward extension of the analysis of [1], provide higher-order long wavelength corrections to the quadratic term

$$|\, v_4\omega^4 + \cdots \,| = O[(\omega L/\bar{c})^4].$$

The wave impedances, which in general are complex functions of frequency, have the long wavelength expansions [2]

$$Z_\pm(\omega) = \pm\bar{\rho}\bar{c}\left\{1 \pm \frac{i}{2}\,\frac{\omega L}{\bar{c}}\,\frac{[\int_0^L \rho(x) \int_0^x m(y)\, dy\, dx - \int_0^L m(x) \int_0^x \rho(y)\, dy\, dx]}{\int_0^L \rho(x)\, dx \int_0^L m(x)\, dx}\right.$$
$$\left. + O\left[\left(\frac{\omega L}{\bar{c}}\right)^2\right]\right\}, \tag{12}$$

where $\bar{\rho}$ is the average density

$$\bar{\rho} = \frac{1}{L}\int_0^L \rho(x)\, dx \tag{13}$$

In the rest of the paper the results already presented are utilized in developing approximations to periodic elastic media. The approximations, which are accomplished by simplifying both the phases and amplitudes of the waves of (4), reduce the long wavelength motion of periodic elastic media to the motion of homogeneous dispersive media.

Amplitude Approximations

As seen from (1) and (2), at the static limit of zero frequency the velocity and stress are independent of position. Accordingly, one can expect that for wavelengths that are sufficiently long the amplitudes $F_\pm(x, \omega)$ and $G_\pm(x, \omega)$ do not depend on x. In order to derive conditions under which the spatial amplitude dependence may be neglected, we consider waves moving in the $+x$ direction (similar arguments apply for the waves moving in the other direction). Because of the amplitude periodicity, we may restrict the analysis to the region in the first unit cell, i.e., $0 \leqslant x \leqslant L$. When the strain is eliminated from (1)–(3) and in the resulting equations expression (4) is substituted, an integration, with the help of (5), of the final result gives

$$F_+(x, \omega) = 1 + i\omega \left[Z_+(\omega) \int_0^x m(y)\, G_+(y, \omega)\, dy - \frac{1}{v_p(\omega)} \int_0^x F_+(y, \omega)\, dy\right], \quad (14)$$

$$G_+(x, \omega) = 1 + i\omega \left[\frac{1}{Z_+(\omega)} \int_0^x \rho(y)\, F_+(y, \omega)\, dy - \frac{1}{v_p(\omega)} \int_0^x G_+(y, \omega)\, dy\right], \quad (15)$$

from which, in conjunction with (9), (10), (12), and (13) follows

$$F_+(x, \omega) = 1 + i\omega(x/\bar{c})\, a_1(x) + O(\omega^2), \quad (16)$$

$$G_+(x, \omega) = 1 + i\omega(x/\bar{c})\, a_2(x) + O(\omega^2), \quad (17)$$

where the functions $a_1(x)$ and $a_2(x)$ are related to the spatial variations in ρ and m as follows

$$a_1(x) = \frac{(1/x)\int_0^x m(y)\, dy - (1/L)\int_0^L m(y)\, dy}{(1/L)\int_0^L m(y)\, dy},$$

$$a_2(x) = \frac{(1/x)\int_0^x \rho(y)\, dy - (1/L)\int_0^L \rho(y)\, dy}{(1/L)\int_0^L \rho(y)\, dy}.$$

The amplitude expansions show that we may neglect the spatial amplitude dependence provided that

$$|(\omega x/\bar{c})\, a_1(x)| \ll 1, \quad (18)$$

$$|(\omega x/\bar{c})\, a_2(x)| \ll 1, \quad (19)$$

where $0 \leqslant x \leqslant L$. The inequalities indicate that when the wavelength λ, $\lambda = 2\pi(\bar{c}/\omega)$, is sufficiently larger than the period L, the spatial amplitude dependence is negligible. However, the length of λ compared to L depends on the spatial variations of ρ and m. In particular, the inequalities show that the stronger the spatial variations of the material properties (i.e., the larger the maximum values of $|a_1(x)|$ and $|a_2(x)|$) the larger the ratio λ/L should be. In the trivial case where the density and inverse modulus are constant the amplitudes are independent of x and λ/L can be arbitrarily small.

When, in addition to (18) and (19), $v_2\omega^2$ is much smaller than 1, i.e.,

$$v_2\omega^2 \ll 1,$$

which is a condition applicable to long wavelengths since from (11) follows that $v_2 < (1/24)(L/\bar{c})^2$, then from (9) and (12) follows that the waves of (4) can be approximated by the following expressions

$$\begin{bmatrix} \hat{u}(x,\omega) \\ \hat{\sigma}(x,\omega) \end{bmatrix} \approx f_+(\omega)\, e^{i(\omega x/v_p(\omega))} \begin{bmatrix} 1 \\ \bar{\rho} v_p(\omega) \end{bmatrix} + f_-(\omega)\, e^{-i(\omega x/v_p(\omega))} \begin{bmatrix} 1 \\ -\bar{\rho} v_p(\omega) \end{bmatrix}. \tag{20}$$

Thus, the long wavelength fields are solutions of the equations

$$\frac{\partial \hat{u}}{\partial x} \approx \frac{i\omega}{\bar{\rho} v_p^2(\omega)}\, \hat{\sigma}, \tag{21}$$

$$\frac{\partial \hat{\sigma}}{\partial x} \approx i\omega\bar{\rho}\hat{u}, \tag{22}$$

which describe motion in a homogeneous dispersive medium of density $\bar{\rho}$ and sound speed $v_p(\omega)$.

Even though the homogeneous dispersive medium equations are simpler than the heterogeneous medium equations, their utility in transient calculations is limited in particular since in most cases, it would be difficult to provide a full description of the frequency dependence of the phase velocity. However, it seems possible to obtain, through experimental or other means, the dominant low frequency behavior of the wave phase velocity. For this reason, further simplications are more appropriate.

Phase Approximations

A particularly simple homogeneous dispersive medium, whose solutions approximate the long wavelength motion in periodic media inside a limited range from the boundary, results when the function $v_p^2(\omega)$ is approximated by its low frequency terms up to second order. Let $\hat{U}(x, \omega)$, $\hat{S}(x, \omega)$, and $\hat{E}(x, \omega)$ represent the velocity, stress and strain in this medium. The motion is governed by the conservation equations

$$\partial \hat{U}/\partial x = -i\omega \hat{E},$$

$$\partial \hat{S}/\partial x = i\omega\bar{\rho}\hat{U},$$

and the constitutive equation [3]

$$\hat{S} = -\bar{\rho}\bar{c}^2(1 - 2v_2\omega^2)\, \hat{E}.$$

Its harmonic solutions are

$$\hat{U}(x, \omega) = f_+(\omega)\, e^{i(\omega x/\bar{c}(1-2v_2\omega^2)^{1/2})} + f_-(\omega)\, e^{-i(\omega x/\bar{c}(1-2v_2\omega^2)^{1/2})}$$
$$\hat{S}(x, \omega) = \bar{\rho}\bar{c}(1 - 2v_2\omega^2)^{1/2} f_+(\omega)\, e^{i(\omega x/\bar{c}(1-2v_2\omega^2)^{1/2})} - \bar{\rho}\bar{c}(1 - 2v_2\omega^2)^{1/2} \times f_-(\omega)\, e^{-i(\omega x/\bar{c}(1-2v_2\omega^2)^{1/2})}.$$

A comparison with Eq. (20) shows that inside the range $x \ll X$, where X is chosen such that the phase difference

$$\left| \frac{\omega X}{v_p(\omega)} - \frac{\omega X}{\bar{c}(1 - 2v_2\omega^2)^{1/2}} \right|$$

is equal to 2π, the homogeneous dispersive medium velocity and stress approximate the velocity and stress of the heterogeneous medium. When all but the dominant terms can be neglected, the range can be found from

$$X = \left| \frac{2\pi\bar{c}}{\omega^5(v_4 - \frac{1}{2}v_2^2)} \right|.$$

By retaining higher-order terms in the expansion for $v_p^2(\omega)$, the range inside which the homogeneous dispersive media are useful for approximating the heterogeneous medium motion can be increased. For example, it is not difficult to show that for the next higher-order dispersive medium, whose constitutive relation is

$$\hat{S} = -\bar{\rho}\bar{c}^2[1 - 2v_2\omega^2 + (v_2^2 - 2v_4)\,\omega^4]\,\hat{E},$$

the range is inversely proportional to ω^7

$$X = |\ \mathrm{const}/\omega^7\ |.$$

Whereas at the limit of zero frequency, the heterogeneous medium generates velocities and stresses that are independent of x, the heterogeneous static strain varies with position according to the detail nature of the heterogeneities. Therefore, the strain $\hat{E}(x, \omega)$, which is independent of position when $\omega = 0$, cannot represent $\hat{\epsilon}(x, \omega)$ accurately. However, $\hat{\epsilon}(x, \omega)$ may be obtained from $\hat{S}$ as follows

$$\hat{\epsilon}(x, \omega) \approx -m(x)\, \hat{S}(x, \omega)$$

provided the detail spatial variation of the modulus is given.

Transient Motion

Like the motion in elastic media, the transient motion of simple dispersive medium, which is governed by the equations

$$U_x = E_t\,, \tag{23}$$
$$S_x = -\bar{\rho} U_t\,, \tag{24}$$
$$S = -\bar{\rho}\bar{c}^2(E + a^2 E_{tt}), \tag{25}$$

where the constant a is equal to $(2v_2)^{1/2}$, has the property, which is desirable for numerical calculations, of being stable. The specific internal energy $\mathscr{E}(x, t)$,

$$\bar{\rho}\mathscr{E}_t = -SU_x,$$

is always positive definite for an initially quiet medium ($E(x, 0) = E_t(x, 0) = 0$) [4]. It is given by

$$\mathscr{E} = \tfrac{1}{2}\bar{c}^2 E + \tfrac{1}{2}\bar{c}^2 a^2 (E_t)^2.$$

In numerical applications, the motion generated in an initially quiet medium is of most interest. To get an idea of what to expect in numerical discretizations of the motion of this medium, we obtain the impulse response. Suppose that a δ-function velocity, $U(0, t) = \delta(t)$, is applied at $x = 0$. Laplace transform theory shows that the fields are given by the expressions

$$U(x, t) = \frac{H(t)}{\pi a} \int_{-1}^{1} \sin(yt') \sin\left(\frac{x'y}{(1 - y^2)^{1/2}}\right) dy + \delta(t)\, e^{-x'}, \tag{26}$$

$$S(x, t) = \bar{\rho}\bar{c}\left[\frac{H(t)}{\pi a} \int_{-1}^{1} (\cos yt' - 1) \cos\left(\frac{x'y}{(1 - y^2)^{1/2}}\right) (1 - y^2)^{1/2}\, dy + \frac{H(t)}{2a}(1 + x')\, e^{-x'} + a\dot{\delta}(t)\, e^{-x'}\right], \tag{27}$$

$$E(x, t) = -\frac{H(t)}{\pi \bar{c} a} \int_{-1}^{1} (\cos yt' - 1) \cos \frac{x'y}{(1 - y^2)^{1/2}} \frac{dy}{(1 - y^2)^{1/2}} - \frac{H(t)}{\bar{c} a} e^{-x'}, \tag{28}$$

where $H(t)$ is the Heaviside function

$$H(t) = 0, \quad t < 0,$$
$$= 1, \quad t > 0,$$

and the normalized variables t' and x' are

$$t' = t/a,$$
$$x' = x/\bar{c}a.$$

The expressions may be verified by substitution into (23)–(25). The following integral (and its spatial derivative) is useful in the verification process [5]

$$\int_0^1 \sin \frac{x'y}{(1 - y^2)^{1/2}}\, y\, dy = \int_0^\infty \left(\frac{\xi}{1 + \xi^2} - \frac{\xi^3}{(1 + \xi^2)^2}\right) \sin(x'\xi)\, d\xi$$
$$= \frac{\pi}{4} x' e^{-x'}.$$

The impulse response of this medium is quite unlike the response of elastic media (case where $a = 0$). The response may be broken up in two parts that depend on their behavior near $t = 0$. The first part, which consists of the integrals appearing in (26)–(28), is zero for $t = 0$. The second part, which consists of the double δ-function, δ-functions, and Heaviside function appearing in (26)–(28), is discontinuous at $t = 0$. The second part shows that, unlike elastic media, this medium is capable of transmitting information instantaneously. However, the information is transmitted with a magnitude that decays exponentially with x with a decay length $\bar{c}a$. Thus, particles whose distance from another particle stationed at, say, x_0 is much larger than $\bar{c}a$ cannot experience significant instantaneous motion based exclusively on what happens at that instant at x_0. In effect, each particle has significant instantaneous communication with neighboring particles that do not lie farther than a distance comparable to $\bar{c}a$. Accordingly, one would expect that numerical instabilities will develop in computations with finite difference analogs of Eqs. (23)–(25) if the spacing between adjacent mesh points is much smaller than $\bar{c}a$.

A Finite Difference Time Marching Analog and Its Stability

A finite difference time marching analog of the differential Eqs. (23)–(25) may be constructed by attaching to difference equations for one-dimensional motion in elastic media an appropriate finite difference analog of the extra term a^2E_{tt}. Although there are many finite difference equations useful for elastic computations, for purposes of restricting the discussion, we adopt the von Neumann–Richtmyer scheme, which is particularly simple and well known [6].

Let the finite difference gridpoints have equal spatial and time spacing Δx and Δt. Let the abbreviated notations $U_m^{n+(1/2)}$, $E_{m+(1/2)}^n$, $S_{m+(1/2)}^n$ stand for $U(x_m, t^{n+(1/2)})$, $E(x_{m+(1/2)}, t^n)$ and $S(x_{m+(1/2)}, t^n)$, where $m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots,$ and $x_m = m\,\Delta x$, $x_{m+(1/2)} = x_m + (\Delta x/2)$, $t^n = n\,\Delta t$, $t^{n+(1/2)} = t^n + \Delta t/2$. The selected difference equations are

$$\frac{U_{m+1}^{n+(1/2)} - U_m^{n+(1/2)}}{\Delta x} = \frac{E_{m+(1/2)}^{n+1} - E_{m+(1/2)}^n}{\Delta t}, \tag{29}$$

$$S_{m+(1/2)}^{n+1} = -\bar{\rho}\bar{c}^2 \left[E_{m+(1/2)}^{n+1} + \left(\frac{a}{\Delta t}\right)^2 (E_{m+(1/2)}^{n+1} - 2E_{m+(1/2)}^n + E_{m+(1/2)}^{n-1})\right], \tag{30}$$

$$\frac{S_{m+(3/2)}^{n+1} - S_{m+(1/2)}^{n+1}}{\Delta x} = -\bar{\rho}\,\frac{U_{m+1}^{n+(3/2)} - U_{m+1}^{n+(1/2)}}{\Delta t}. \tag{31}$$

At the start of each computational cycle, $U_m^{n+(1/2)}$ and all other quantities of superscript n or less are known. To start computations, we find, for each m, $E_{m+(1/2)}^{n+1}$

from (29). Then, we obtain $S^{n+1}_{m+(1/2)}$, for each m, from (30). To finish the cycle, we compute, for each m, $U^{n+(3/2)}_{m+1}$ from (31). To compute motion generated by a boundary velocity $v(t)$, the variables $U^{n+(1/2)}_0$ were assigned the values $v(t^{n+(1/2)})$. The variables $U^{(1/2)}_{m+1}$, $E^0_{m+(1/2)}$, $E^{-1}_{m+(1/2)}$, and $E^{-2}_{m+(1/2)}$ were assigned zero values.

The stability of time marching computations can be examined by considering solutions of the form

$$U^{n+(1/2)}_m = d_1(K)\, e^{imK\Delta x} e^{-i(n+(1/2))\omega(K)\Delta t},$$

$$S^n_{m+(1/2)} = d_2(K)\, e^{i(n+(1/2))K\Delta x} e^{-in\omega(K)\Delta t},$$

$$E^n_{m+(1/2)} = d_3(K)\, e^{i(m+(1/2))K\Delta x} e^{-in\omega(K)\Delta t}.$$

By substitution into Eqs. (29)–(31), we find that the wave number K and frequency ω were related by the dispersion relation

$$\sin^2 \frac{K\Delta x}{2} = \left(\frac{\Delta x}{\bar{c}\,\Delta t}\right)^2 \frac{\sin^2(\omega\,\Delta t/2)}{1 - 4(a/\Delta t)^2\, e^{i\omega\Delta t} \sin^2(\omega\,\Delta t/2)}. \tag{32}$$

The values of the growth factor $e^{-i\omega\Delta t}$ are obtained from the roots of (32). When the phase $\omega\,\Delta t$ is spit into real and imaginary parts, $\omega\,\Delta t = \varphi - i\psi$, Eq. (32) gives the following relations between φ and ψ,

$$\sin\varphi \sinh\psi = \frac{8\alpha^2\beta^2 e^{\psi} \sin\varphi}{|\,1 + 4\alpha^2 e^{\psi} e^{i\varphi}\,|^2}, \tag{33}$$

$$\cos\varphi \cosh\psi = 1 - 2\beta^2 \frac{1 + 4\alpha^2 e^{\psi} \cos\varphi}{|\,1 + 4\alpha^2 e^{\psi} e^{i\varphi}\,|^2}, \tag{34}$$

where

$$\alpha^2 = \left(\frac{\bar{c}a}{\Delta x}\right)^2 \sin^2 \frac{K\Delta x}{2},$$

$$\beta^2 = \left(\frac{\bar{c}\,\Delta t}{\Delta x}\right)^2 \sin^2 \frac{K\Delta x}{2}.$$

For stability, we require that the roots of Eqs. (33) and (34) have nonnegative ψ. When $\sin\varphi \neq 0$, only positive values of ψ satisfy Eq. (33). Thus, all complex growth factors have magnitudes less than 1. Moreover, it is not difficult to show from Eq. (34) that the growth factor cannot be real and positive. The magnitude of negative growth factors is governed by Eq. (34), where $\varphi = \pi$ and can be obtained from

$$\cosh\psi = -1 + \frac{2\beta^2}{1 - 4\alpha^2 e^{\psi}}.$$

The roots of this equation occur for values of ψ for which the right-hand side is greater than or equal to 1. However, it is not difficult to show that the right-hand side is greater than 1 only when e^{ψ} lies between $(1 - \beta^2)/4\alpha^2$ and $1/4\alpha^2$. Thus, for all roots, the values of ψ are such that

$$\frac{1 - \beta^2}{4\alpha^2} \leqslant e^{\psi} \leqslant \frac{1}{4\alpha^2}.$$

Clearly, the roots will have ψ positive if $(1 - \beta^2)/4\alpha^2 > 1$. Hence, in all cases, the time marching computations are stable if

$$4\left(\frac{\bar{c}a}{\Delta x}\right)^2 + \left(\frac{\bar{c}\,\Delta t}{\Delta x}\right)^2 < 1. \tag{35}$$

When a is equal to zero, Eq. (35) gives the familiar condition for elastic equations. However, for nonzero values of a, stability cannot be ensured with zone width smaller than $2\bar{c}a$. Thus, the scheme is useful only in computations of long wavelength motion.

Phase Velocity of Finite Difference Waves and Limitations Placed on the Mesh Variables

The analytical solution of the finite difference equations for a velocity boundary condition may be investigated by separation of variable techniques. The fields may be written as

$$U_m^{n+(1/2)} = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} A(\omega)\, e^{iK(\omega)m\,\Delta x} e^{-i(n+(1/2))\omega(2\pi/\tau)}\, d\omega,$$

$$S_{m+(1/2)}^n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} Y(\omega)\, A(\omega)\, e^{iK(\omega)(m+(1/2))\,\Delta x} e^{-in\omega(2\pi/\tau)}\, d\omega,$$

$$E_{m+(1/2)}^n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} W(\omega)\, A(\omega)\, e^{iK(\omega)(m+(1/2))\,\Delta x} e^{-in\omega(2\pi/\tau)}\, d\omega,$$

where

$$\tau = 2\pi/\Delta t,$$

$$A(\omega) = \sum_{r=0}^{\infty} v(t^{r+(1/2)})\, e^{i(r+(1/2))\omega(2\pi/\tau)}.$$

The wave number K and functions Y and W are obtained by substitution in (29)–(31). The wave number K and frequency ω are related by the previously obtained dispersion relation (32). A low frequency investigation of this relation shows that the phase velocity $V_p(\omega)$, $V_p(\omega) = \omega/K(\omega)$, of the finite difference waves has the expansion

$$V_p(\omega) = \bar{c}\left\{1 - \frac{1}{2}\left(a^2 + \frac{1}{12}\left[\left(\frac{\Delta x}{c}\right)^2 - (\Delta t)^2\right]\right)\omega^2 - i\frac{a^2}{2}\Delta t\,\omega^3 + O(\omega^4)\right\}. \quad (36)$$

A comparison of this expansion and Eq. (9) indicates that, for calculations of long wavelength motion in periodic elastic media, more limitations than the stability conditions must be placed on Δx and Δt.

a. *Limitations Placed on* Δt

The high-order cubic frequency term in the low frequency phase velocity expansion creates energy losses [7]. Evidently, the dissipation has been introduced from the desired (for starting the computations from zero initial conditions, and for making the scheme explicit) backward-in-time representation of a^2E_{tt}. When Δt is sufficiently small, this term, which is the dominant term responsible for dissipation in the long wavelength part of the spectrum, can be made negligible at long wavelengths. An estimate of how small Δt should be may be obtained from the long wavelength expression for the propagation factor λ, $\lambda = e^{iKx}$, where $x = m\,\Delta x$. From Eq. (36), we find that

$$K = \frac{\omega}{\bar{c}} + \frac{1}{2}\left\{a^2 + \frac{1}{12}\left[\left(\frac{\Delta x}{\bar{c}}\right)^2 - (\Delta t)^2\right]\right\}\frac{\omega^3}{\bar{c}} + i\frac{a^2\,\Delta t}{2\bar{c}}\,\omega^4 + O(\omega^5).$$

Thus, in the long wavelength part of the spectrum, the propagation factor may be approximated as follows

$$\lambda \approx e^{i(\omega/\bar{c})x}e^{(i/2)\{a^2+(1/12)[(\Delta x/\bar{c})^2-(\Delta t)^2]\}(\omega^3/\bar{c})x}e^{-(a^2\Delta t/2\bar{c})\omega^4 x}.$$

Therefore, the long wavelength dissipation will be small when, for $|\omega| \ll 1/a$, the exponential term is close to unity, i.e., for Δt small enough that

$$\Delta t \ll 2\bar{c}a^2/x$$

b. *Limitations Placed on* Δx

The quadratic coefficient in the low frequency phase velocity expansion contains a term that goes to zero as $a \to 0$ and a term independent of a. When Δt is small, the latter term, which arises from the low-order spatial discretizations employed by the Neumann–Richtmyer scheme [8], has minimum effect provided that Δx is as small as possible. Accordingly, in order that the second-order dispersion of the von Neumann–Richtmyer scheme does not mask the second-order dispersion of the heterogeneous medium waves, the mesh size must be close to the stability limit $2\bar{c}a$.

Example

To generate an example that gives an idea of the accuracy of the computational model in calculations of long wavelength motion, the results for a velocity boundary condition were compared to the known exact solution of a periodic bilaminate [9]. The first bilaminate layer starts at $x = 0$, is 0.8 cm wide, has a density of 1 g/cm^3, and a sound speed of 1 cm/μsec. The second layer, which is 0.2 cm wide, is weaker than the first. Its density and sound speed have the values 0.5 g/cm^3 and 0.5 cm/μsec, respectively. Beyond the second layer, the bilaminate repeats itself with period 1 cm. The values of the parameters $\bar{\rho}$, $\bar{c}$, and a are, respectively, 0.9 g/cm^3, 0.68041 cm/μsec, and 0.2357 μsec. The boundary velocity is a square pulse of height U_0 cm/μsec and duration 7.2 μsec. The time step and zone size have the values 0.005 μsec and 0.35 cm, respectively.

Figures 1 and 2 display the calculational and exact particle velocity and stress profiles that develop at positions $x = 35$ cm and $x = 7$ cm. The figures show that the computational model is capable of reproducing some of the effects of the heterogeneities on the square pulse. In particular, the wave front erosion, which is generated from repeated reflections from the bilaminate interfaces, and the main low frequency wave body are reproduced rather accurately. However, it is evident that the more rapid oscillations, which follow the main wave body and are heavily attenuated, have incorrect phasing. Thus, it seems probable that a more accurate reproduction of these oscillations would have to employ a nondissipative scheme that provides higher-order low frequency approximations to the wave phase velocity.

The profiles at $x = 1$ cm displayed in Fig. 3 show an interesting phenomenon. Whereas the particle velocity is reproduced satisfactorily, the stress is not. It is evident, however, that in contrast to the exact stress profiles at 7 cm and 35 cm, the exact stress profile at 1 cm contains strong high frequency components. The nature of these high frequencies has been explained in [9]. It turns out that the near boundary behavior of the velocity and stress can be dominated by resonance phenomena that occur at frequencies at which the wave impedance $Z_+(\omega)$ defined in Eq. (8) has poles (or zeros if a stress boundary condition is used). To obtain a limited understanding of the resonance condition in an arbitrary periodic medium one may utilizer a similar method to the one used in [9]. In particular, with this analysis one can show from Eq. (8) that all resonances can occur either at frequencies where the group velocity $v_g(\omega)$, $v_g(\omega) = \partial\omega/\partial k$ where $k = \omega/v_p$, is complex, or at frequencies where $\sin kL = 0$ [10]. Although this analysis does not rule out the possibility of some resonances occurring at frequencies where $\sin kL = 0$, and where it happens that the group velocity is real and nonzero, nevertheless, no calculation or experiment has provided evidence for propagating resonance phenomena. Accordingly, from the possible resonance locations mentioned above,

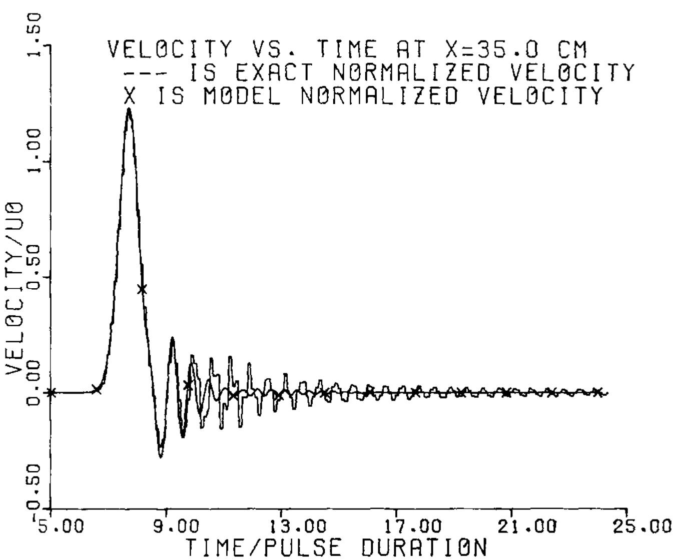

FIGURE 1a

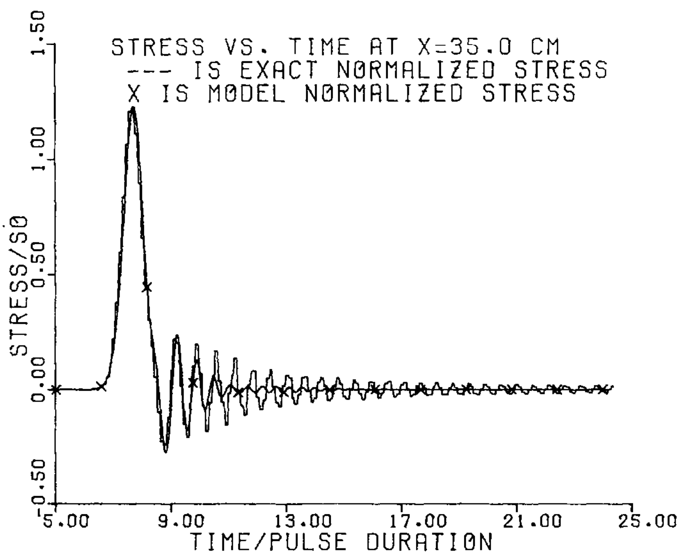

FIGURE 1b

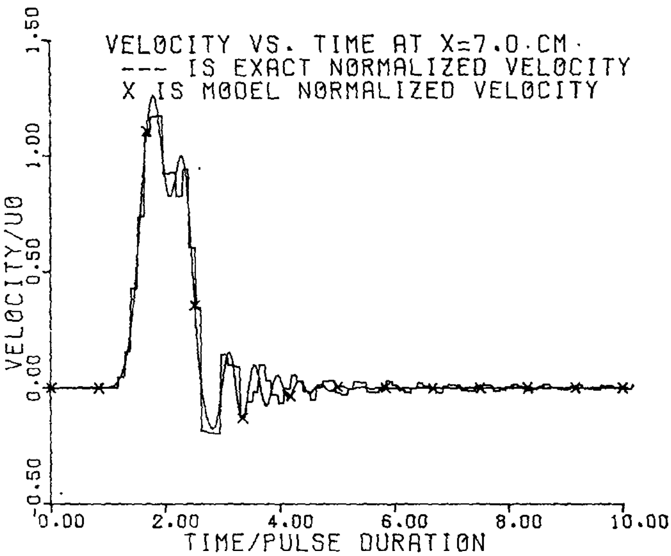

FIGURE 2a

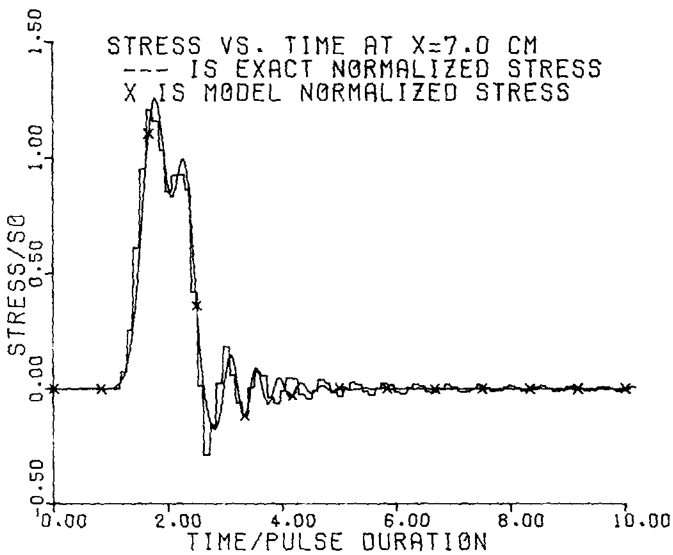

FIGURE 2b

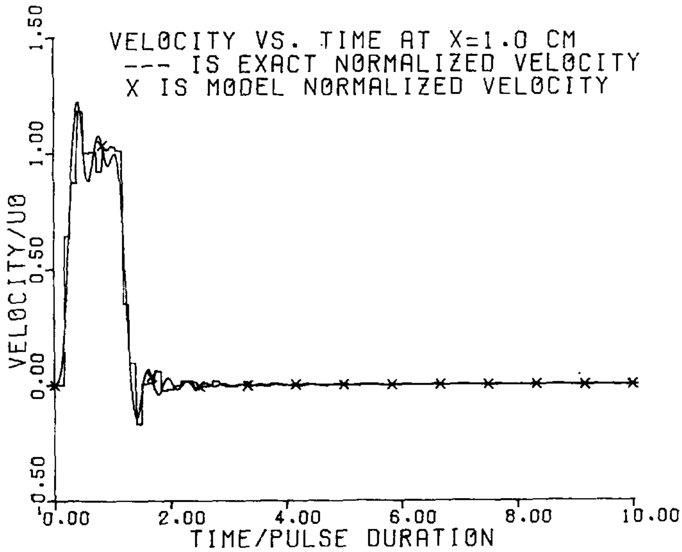

FIGURE 3a

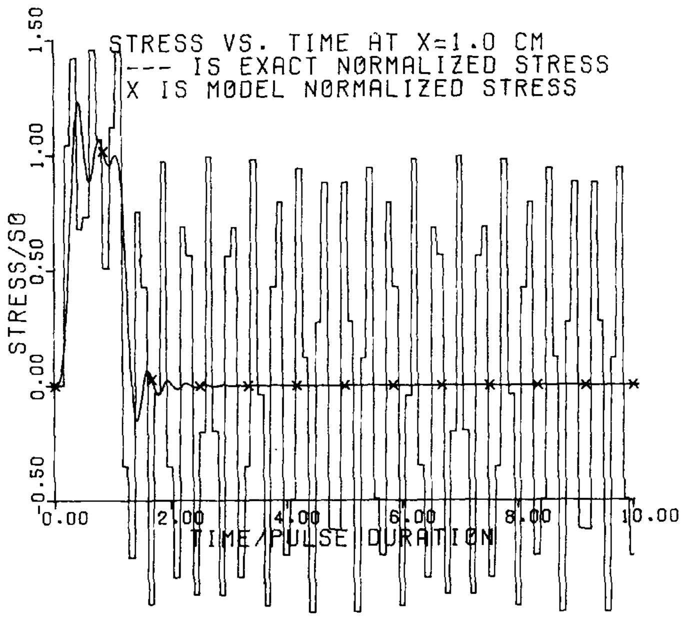

FIGURE 3b

one would expect that only frequencies where the group velocity is complex, or possibly zero, can resonate. Therefore, the resonance phenomena should decay very rapidly with distance without influencing the solution at ranges whose distance from the boundary is larger than a few periods.

CONCLUSIONS

The uniaxial motion of heterogeneous periodic elastic media was investigated. It was shown that for long wavelengths, the harmonic motion of any periodic medium can be approximated by the motion of suitably chosen homogeneous dispersive media. The simplest of these media has a constitutive relation that contains, in addition to the usual Hookean term, an extra term that is proportional to the second time derivative of the strain. The proportionality constant is related to the dominant coefficient of the low frequency power series expansion of the phase velocity of the periodic medium.

The transient properties of this simple medium were examined and a particular set of finite difference equations, which are based on the von Neumann–Richtmyer scheme, was analyzed. Stability was investigated with the von Neumann–Richtmyer method. It was shown that for the scheme to be stable, the mesh size cannot be smaller than a certain characteristic length that is proportional to the coefficient of the extra term in the constitutive relation. Particular choices of Δx and Δt that are useful for modeling long wavelength propagation in periodic media were obtained by examining the low frequency dispersion characteristics of the finite difference waves. An application of the difference equations to calculations of transient motion was given.

ACKNOWLEDGMENTS

This work was supported in part by the Defense Atomic Support Agency under Contract DASA01-70-C-0074. The author would like to thank Dr. John G. Trulio of Applied Theory, Inc., Professor P. Saffman of the California Institute of Technology, Dr. Michael El Raheb of Tetra Tech, Inc., and reviewers of this and other journals for their comments on various aspects of the problem that were given an incomplete treatment in earlier drafts of this paper.

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1. G. N. BALANIS, *J. Math. Phys.* **16** (1975), 1383.
2. Although these expansions are not explicitly given in [1], nevertheless, they can be easily obtained since they depend on quantities whose expansions were included in this reference.
3. It is noted that there are many ways of writing the constitutive relation for the purpose of obtaining the desired phase velocity low frequency dependence. The way chosen is in direct

analogy to the case of viscoelastic media. When there is dissipation v_p has, in general, the low frequency expansion

$$v_p = \bar{c}(1 - iv_1\omega - v_2\omega^2 - iv_3\omega^3 - \cdots),$$

where v_1, v_2, v_3,... are real constants. When v_p^2 is approximated by its dominant low frequency terms,

$$v_p^2 \approx \bar{c}^2(1 - 2iv_1\omega),$$

the well-known constitutive relation of viscoelasticity is obtained.

4. Similar remarks apply to the three-dimensional dispersive media, whose constitutive equation is

$$S_{ij} = -\bar{\rho}\{(c_l^2 - 2\bar{c}_t^2)\delta_{ij}E_{kk} + 2\bar{c}_t^2 E_{ij}\}$$
$$-\bar{\rho}\left\{(a_l^2\bar{c}_l^2 - 2a_t^2\bar{c}_t^2)\delta_{ij}\frac{\partial^2}{\partial t^2}E_{kk} + 2a_t^2\bar{c}_t^2\frac{\partial^2}{\partial t^2}E_{ij}\right\},$$

where α_l and α_t are the longitudinal and transverse dispersion constants, which may be useful in modeling the long wavelength propagation in heterogeneous media exhibiting a homogeneous and isotropic macroscopic elastic behavior.

5. I. S. Gradshteyn and I. M. Ryzhik, "Table of Integrals, Series and Products," p. 406, 410. Academic Press, New York, 1965.
6. J. Von Neumann and R. D. Richtmyer, *J. Appl. Phys.* **21** (1950), 232.
7. It may be worthwhile to note that the third-order dissipation may make attractive the inclusion of the term $\alpha^2\epsilon_{tt}$ in some calculations with the von Neumann–Richtmyer scheme, where dissipation of order higher than the first is desired. In such application, the choice of a as a small parameter tied to the mesh size, say

$$a^2 = \alpha(\Delta x/\bar{c})^2,$$

where α is a positive constant less than 0.25, may be useful.

8. It may be worthwhile to note that, for zero Δt, the characteristic relation of the von Neumann–Richtmyer scheme (Eq. (32), where $a^2 = 0$, $\Delta t = 0$) reduces to the dispersion relation for longitudinal oscillations in a system of equal masses and springs. Accordingly, alternative numerical applications that do not generate any dissipation and are not tied to the homogeneous dispersive medium may be obtained by utilizing the dispersion properties of the von Neumann–Richtmyer scheme without introducing the term $a^2\epsilon_{tt}$. In such applications, one could use special choices of Δx and Δt that are attractive (for reasons of economy or other particular reasons) and satisfy (within the stability limit of the von Neumann–Richtmyer scheme) the equation

$$(\Delta x/\bar{c})^2 - (\Delta t)^2 = 12a^2.$$

9. G. N. Balanis, *J. Appl. Mech.* **40** (1973), 815.
10. As seen from Eq. (8), when ω corresponds to a pole or zero of $Z_+(\omega)$, then $b_{12}(L, \omega) = 0$ or $b_{21}(L, \omega) = 0$. However, since the determinant of matrix $B(L, \omega)$ is equal to 1, it follows that at the resonances $b_{11}(L, \omega)b_{22}(L, \omega) = 1$. This result and Eq. (7) show that

$$|\cos kL| \geqslant 1.$$

However, the group velocity is given by

$$v_g(\omega) = \frac{2L(1 - \cos^2 kL)^{1/2}}{(\partial/\partial\omega)[b_{11}(L, \omega) + b_{22}(L, \omega)]}.$$

Thus, at the resonant frequencies at which $\sin kL \neq 0$, the group velocity is complex.